Announcements

1) Stephen DeBacker - student presentation, 4-5, this room

2) Last HW - due Thursday next week.

3) Error on notes –
will now be corrected!

Theorem: Every matrix $A \in \mathbb{C}^{m \times m}$ has a Schur factorization

$$A = Q T Q^*$$

with Q unitary and T upper triangular.

begin "proof"

2×2: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

we know there is an eigenvalue λ for A with corresponding eigenvector v.

Want a unitary Q_1 with

$$Q_1^* A Q_1 = \begin{bmatrix} \lambda & e \\ 0 & f \end{bmatrix}$$

We pick the eigenvalue v to be of unit length, then let Q_1 be the unitary that maps

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to v.

Then $A = Q_1 \begin{bmatrix} \lambda & e \\ 0 & f \end{bmatrix} Q_1^*$.

We have $Q_1\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = v$

and $Q_1\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \in \{v\}^{\perp}$.

Consider

$$e_1^* \cdot \underbrace{Q_1^* A Q_1 e_1}$$

$$= Q_1^* A v$$

$$= Q_1^* \lambda v$$

$$= \lambda Q_1^* v$$

$$= \lambda e_1$$

Therefore,

$$e_1^* \cdot Q_1^* A Q_1 e_1 = e_1^* \lambda e_1$$
$$= \lambda$$

This shows the (1,1) entry of $Q_1^* A Q_1$ is λ.

The $(2,1)$ entry is

$$e_2^* Q_1^* A Q_1 e_1$$

$$= (Q_1 e_2)^* (A Q_1 e_1)$$

$$= (Q_1 e_2)^* (A v)$$

$$= (Q_1 e_2)^* \lambda v$$

$= 0$ since $Q_1 e_2$ is orthogonal to v.

This shows

$$Q_1^* A Q_1 = \begin{bmatrix} \lambda & e \\ 0 & f \end{bmatrix}$$

$$= T \text{, upper-triangular}$$

So

$$A = Q_1 T Q_1^* .$$

3x3 case

Send A to $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$.

Apply the unitary Q_1,

where if v is an eigenvector of unit length for eigenvalue λ_1

$$Q_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = v.$$

$$Q_1^* A Q_1 = \begin{bmatrix} \lambda & h & i \\ 0 & j & k \\ 0 & l & m \end{bmatrix}$$

Apply the unitary

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & R \end{bmatrix}$$

where R is the unitary that maps $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to an eigenvector of unit length of $\begin{bmatrix} j & k \\ l & m \end{bmatrix}$.

Then

$$Q_2^* Q_1^* A Q_1 Q_2 = \begin{bmatrix} \lambda & a & b \\ 0 & c & d \\ 0 & 0 & e \end{bmatrix}$$

and you'll get, with

$$Q = Q_1 Q_2,$$

$$A = Q \begin{bmatrix} \lambda & a & b \\ 0 & c & d \\ 0 & 0 & e \end{bmatrix} Q^*$$

$$= Q T Q^*.$$

Algorithms

Some dilemmas

1) Backwards Stability is not even even a hope initially since finding the roots of a polynomial is an ill-conditioned problem.

2) Degree bigger than 5:

no formula involving square roots, addition, multiplication, etc. can even possibly exist! (Abel, Galois)

Solution: Consider iterative algorithms that converge to the eigenvalues of a given matrix.

Definition: (Hessenberg matrix)

A matrix $A \in \mathbb{C}^{m \times m}$ is called **Hessenberg** if

$A_{i,j} = 0$ whenever

$i \geq j+2$.

(upper triangular: $i \geq j+1$)

Example 1:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 9 & 10 & 11 \\ 0 & 0 & 12 & 13 \end{bmatrix}$$

is Hessenberg.

2 - Pronged Approach

1) Compute Hessenberg matrix H_1 "associated" to A.

2) Perform an iterative algorithm on H_1 to get $(H_n)_{n=2}^{\infty}$ Hessenberg

What does $(H_n)_{n=1}^{\infty}$ do?

If $A = QTQ^*$,

the Schur decomposition,

then $H_n \to T$

as $n \to \infty$.

Recall: (upper-triangular + self-adjoint)

If A is upper-triangular and self-adjoint, then A is diagonal.

Definition: (tridiagonal matrix)

A matrix $A \in \mathbb{C}^{m \times m}$ is **tridiagonal** if

$$A_{i,j} = 0 \quad \text{whenever}$$

$$|j - i| > 1.$$

Example 2:

$$A = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 16 & 2 & 7 & 0 & 0 \\ 0 & 11 & 3 & 8 & 0 \\ 0 & 0 & 12 & 4 & 9 \\ 0 & 0 & 0 & 13 & 5 \end{bmatrix}$$

is tridiagonal.

Observation: (Hessenberg & self-adjoint)

Let $A \in \mathbb{C}^{m \times m}$ be Hessenberg and self-adjoint. Then A is tridiagonal.

Stage 1 of the Algorithm

Householder reflections, but not applied to all of A! We choose our first unitary in the form

$$\begin{bmatrix} 1 & 0 \text{ --- } 0 \\ 0 & \\ \vdots & \tilde{Q}_1 \\ 0 & \end{bmatrix}$$

where $\tilde{Q}_1$ sends e_1^{m-2} to an eigenvector of the bottom right submatrix of A.

Algorithm :

(Matlab)